USEFUL HINTS FOR SOLVING PHYSICS OLYMPIAD PROBLEMS

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Introduction:

Physicists often attempt to solve difficult problems with simple methods. There are several reasons for doing this:

It's faster: the simple method provides a quick approximation to a more complete solution that could be obtained with a more complicated method.

You have no choice: the problem might be too difficult to solve without a simplification.

Another perspective: the problem has already been solved and we want to solve it again with a simple method, both to check our answer and to see if we can gain some additional insight into the solution.

What I'd like to talk about today are some of the simple methods that you might use to solve physics problems. In particular, there are three main ideas that I would like to discuss:

(1) Dimensional analysis
(2) Oscillations
(3) Approximations

Some of this material was also given in a similar talk at the Physics Olympiad Weekend at UBC in 2006 and the transcript of that talk is also available on the web.

First Example:

Consider the vertical oscillations of a mass hanging from the end of a spring. Without a direct measurement of the oscillation period, is there a way that we could predict this? Sure, there must be some sort of formula: $T = ?$ Even if we don't know what this formula is, it turns out that we can guess it.

The first step is to determine the variables that T might depend on. It seems reasonable to expect that T might depend on as many as four variables: m, the mass which hangs from the spring; k, the spring stiffness constant; A, the amplitude of the oscillations; and g, the acceleration due to gravity.

In the second step, we guess that the formula involves only multiplication and division of various powers of our variables,

$$T = Cm^{\alpha} k^{\beta} A^{\gamma} g^{\delta}$$

where C is a dimensionless constant and $\alpha, \beta, \gamma,$ and δ are unknown exponents.

The third step involves examining the units which appear in our formula. All correct equations in physics have consistent units, therefore we require

$$[\mathrm{s}] = [1][\mathrm{kg}]^{\alpha} \left[\frac{\mathrm{N}}{\mathrm{m}}\right]^{\beta} [\mathrm{m}]^{\gamma} \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]^{\delta}$$

We can replace the N with $\mathrm{kg\ m/s^2}$ (think of $F = ma$) so that we have

$$[\mathrm{s}] = [\mathrm{kg}]^{\alpha} \left[\frac{\mathrm{kg}}{\mathrm{s}^2}\right]^{\beta} [\mathrm{m}]^{\gamma} \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]^{\delta} = [\mathrm{kg}]^{\alpha+\beta} [\mathrm{m}]^{\gamma+\delta} [\mathrm{s}]^{-2\beta-2\delta}$$

This leads to three equations, one for each unit:

$$\mathrm{s} \;\Rightarrow\; 1 = -2\beta - 2\delta$$
$$\mathrm{kg} \;\Rightarrow\; 0 = \alpha + \beta$$
$$\mathrm{m} \;\Rightarrow\; 0 = \gamma + \delta$$

At this stage, we have a problem: 3 equations involving 4 unknowns do not lead to a unique solution, so at this stage we would conclude that there are many possible ways to write a formula for the oscillation period that has consistent units.

Suppose we do a quick bit of experimentation to see how the oscillations change if the spring is not vertical. It can quickly be seen that the oscillations have the same period if the spring is vertical, horizontal, or at any angle in between. This means that the oscillation period does not depend on g and therefore that $\delta = 0$ in the equations above. This leaves us with 3 equations for 3 remaining unknowns,

$$\mathrm{s} \;\Rightarrow\; 1 = -2\beta$$
$$\mathrm{kg} \;\Rightarrow\; 0 = \alpha + \beta$$
$$\mathrm{m} \;\Rightarrow\; 0 = \gamma$$

These are easily solved: $\beta = -\tfrac{1}{2}$, $\alpha = -\beta = +\tfrac{1}{2}$, and $\gamma = 0$. Substituting these exponents into the original formula for T, we have

$$T = Cm^{1/2} k^{-1/2} A^0 g^0 = C\sqrt{\frac{m}{k}}$$

This technique of guessing a formula by seeing if there is a unique way for the units to work out is known as **dimensional analysis**. In the example considered here, we have shown that the oscillation period cannot depend on the amplitude of the oscillations. This technique cannot tell us about the dimensionless constant C, however. Only with an explicit derivation using the laws of physics can we determine that $C = 2\pi$ so that

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Explicit Derivation:

Where does the 2π come from in the formula for the oscillation period? We'll look at two different ways of showing this. The first method involves calculus, and while calculus is not required to solve Physics Olympiad problems, it is allowed. The starting point is Newton's second law, $\Sigma F = ma$. According to Hooke's Law, the spring exerts a force $F = -kx$ on an attached mass, where k is the spring stiffness constant, x is the displacement of the mass from the equilibrium point, and the minus sign indicates that the force will always be directed back towards the equilibrium point. We will ignore gravity here, since it merely changes the exact location of the equilibrium point without affecting the oscillation period. This leads to

$$-kx = ma$$

and writing acceleration as the second derivative of displacement,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The motion of the mass is given by the solution of this differential equation:

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

(This can be checked by differentiation. Note that cosine could also be used.) Working in radians, the sine function repeats itself with a period of 2π, therefore the motion of the mass will repeat itself after a time T given by the condition

$$\sqrt{\frac{k}{m}}\,T = 2\pi \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Without calculus, we need to know that the Hooke's Law force $F = -kx$ is associated with a potential energy $U(x) = \frac{1}{2}kx^2$. (Technically this result is also obtained via calculus, as $U(x) = -\int F(x)dx$, but this particular formula for $U(x)$ can be found as the area of a triangle under the graph of $-F(x)$ vs. x.) The total energy of the oscillating spring-mass system – kinetic and potential – is conserved, therefore

$$\tfrac{1}{2}mv_{\mathrm{MAX}}^2 = \tfrac{1}{2}kA^2$$

since on the LHS, all of the system's energy is kinetic at the equilibrium point while on the RHS, all of the system's energy is potential at the turnaround points. Next, we recognize the one-dimensional oscillations of the mass on a spring as the "shadow" of two-dimensional uniform circular motion whereby the consideration of one complete circle leads to

$$v_{\mathrm{MAX}} = \frac{2\pi A}{T} \quad \Rightarrow \quad T = 2\pi\frac{A}{v_{\mathrm{MAX}}}$$

The energy conservation equation allows us to replace the ratio A/v_{MAX} with $\sqrt{m/k}$ so that we once again obtain the correct formula for the oscillation period:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

A Useful Approximation:

In anticipation of the next example, let's step away from physics in order to construct a very important mathematical approximation. Consider the following simple algebraic identities:

$$(1+x)^2 = 1 + 2x + x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

If *x* is much less than one (or, if *x* is negative, $|x| << 1$) then x^2 is smaller still and it might not be a bad approximation to ignore it (and even higher powers of *x*) altogether. This leads to

$$(1+x)^2 \cong 1 + 2x$$
$$(1+x)^3 \cong 1 + 3x$$
$$(1+x)^4 \cong 1 + 4x$$

The pattern is obvious:

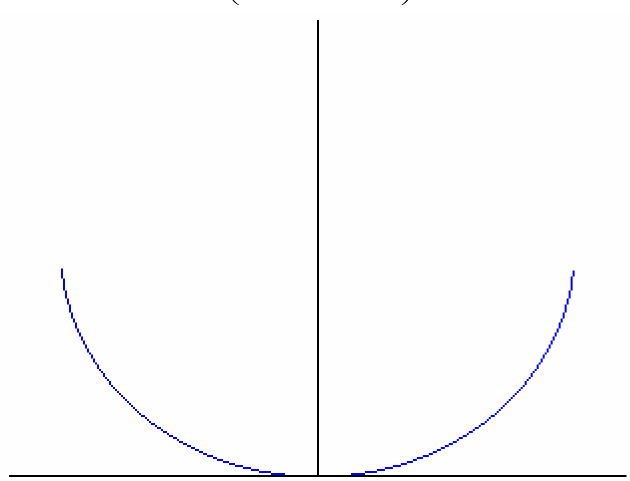
$$(1+x)^n \cong 1 + nx \qquad \text{for } |x| << 1$$

It turns out that this approximation is valid for *any* value of *n*, not just whole numbers. This is known as the **binomial approximation**.

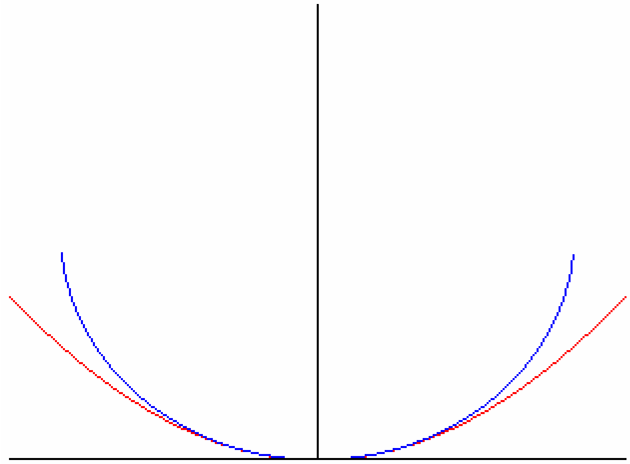
Second Example:

Perhaps the most familiar oscillating system is the simple pendulum, where a small bob of mass *m* swings at the end of a light string of length *L*. As an exercise, you can confirm that dimensional analysis on the variables *L*, *g*, and *m* leads to the formula $T = C\sqrt{L/g}$. We will now obtain this result in a different way. Since the pendulum bob moves back and forth in arcs of a circular path, we can easily sketch the potential energy function $U(x) = mgy(x)$ by taking the bottom half of the circle $x^2 + (y-L)^2 = L^2$ so that

$$U(x) = mg\left(L - \sqrt{L^2 - x^2}\right)$$



Near the bottom, the circle can be approximated by a parabola:



This means that the potential energy function is approximately $U(x) = \frac{1}{2}kx^2$ when x is near zero. If we can determine the value of k which provides the match, we will be able to determine the period of the pendulum from $T = 2\pi\sqrt{m/k}$. We do this by algebraically approximating the exact expression for the potential energy:

$$\begin{aligned} U(x) &= mg\left(L - \sqrt{L^2 - x^2}\right) \\ &= mgL\left(1 - \sqrt{1 - (x^2/L^2)}\right) \\ &\cong mgL\left(1 - \left\{1 - \tfrac{1}{2}(x^2/L^2)\right\}\right) \\ &\cong mgL\left(\frac{x^2}{2L^2}\right) \\ &\cong \frac{1}{2}\left(\frac{mg}{L}\right)x^2 \end{aligned}$$

Note that the binomial approximation (with $n = \frac{1}{2}$ and $"x" = -\frac{1}{2}(x^2/L^2)$) was used in the third line under the assumption that $(x^2/L^2) << 1$. This allows us to identify the k value needed to find the period of the pendulum:

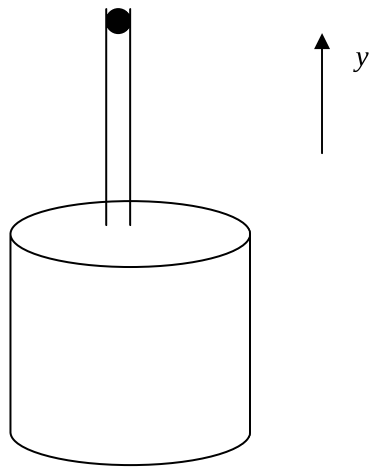
$$k = \frac{mg}{L} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{(mg/L)}} = 2\pi\sqrt{\frac{L}{g}}$$

General Oscillations:

The method used in the previous example is applicable to any kind of oscillating system, particularly when the oscillations have a small amplitude. To reiterate, we are looking to find a k value for the oscillating system, using either **(1)** forces, $F = -kx$, or **(2)** potential energy, $U = \frac{1}{2}kx^2$, where x is the displacement of the system from a point of stable equilibrium. Once we have determined the k value, the oscillation period is simply $T = 2\pi\sqrt{m/k}$.

Third Example:

Consider a large glass jar with a tall, thin tube projecting upwards from the top (see the sketch below). If a ball bearing, whose diameter is such that it just barely fits into the tube, is dropped into the tube, gravity will pull the ball down the tube. Then a strange thing happens: the ball slows down, stops, and bounces back up towards the top of the tube whereby it continues to oscillate up and down.



This occurs because air cannot go around the ball from one side of the tube to the other, therefore the air in the jar is compressed as the ball falls. Air pressure inside the jar is increased as a result and this leads to an upward force on the ball. If we denote the ball's position in the tube by a coordinate *y* which is zero at the top and points upward, it can be shown that the net force acting on the ball is

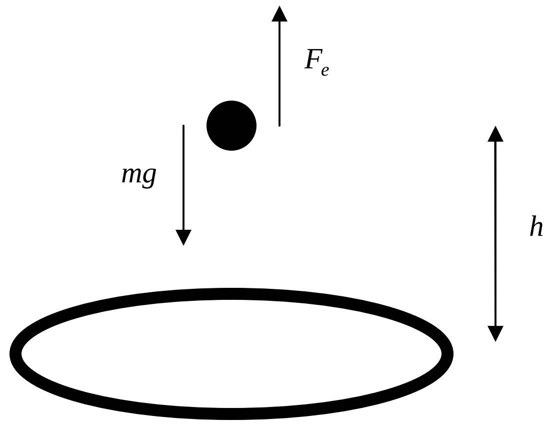
$$F(y) = -\frac{\gamma P_{\text{atm}} A^2}{V} y - mg$$

where: $\gamma \cong 1.4$ is a constant which relates to the details of how easily air can be compressed; P_{atm} is atmospheric pressure; *A* is the cross-sectional area of the tube; *V* is the volume of air in the jar; and *m* is the mass of the ball. This force equation is of the form $F = -k(y - y_0)$, therefore we can determine the period of the oscillations:

$$k = \frac{\gamma P_{\text{atm}} A^2}{V} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mV}{\gamma P_{\text{atm}} A^2}}$$

Fourth Example:

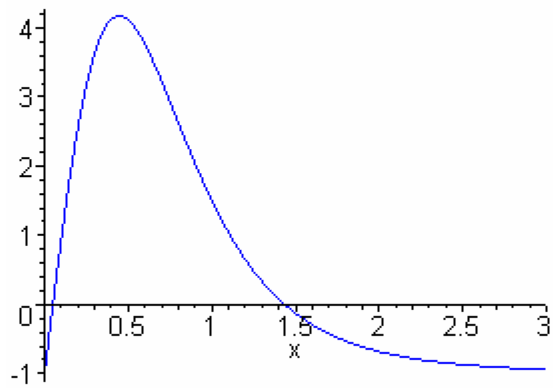
In the fourth question of the 2008 selection exam, an electrically charged dust particle was suspended above an electrically charged ring:



Based on the net force

$$F(h) = \frac{kQqh}{(r^2 + h^2)^{3/2}} - mg$$

it was determined that there are two equilibrium points where $F = 0$: the first is close to the ring and is unstable while the second is further from the ring and is stable (in the vertical direction). This is shown in the graph below (where $x = h/r$).



If the particle is at the stable equilibrium point and is gently nudged, it will oscillate about the equilibrium point. Writing

$$h = (h - h_{\text{EQM}}) + h_{\text{EQM}} = y + h_{\text{EQM}}$$

it is left as a (challenging) exercise to show that the net force can be written as

$$F(y) \cong -\frac{kQq(2h_{\text{EQM}}^2 - r^2)}{(r^2 + h_{\text{EQM}}^2)^{5/2}}\, y$$

For the stable equilibrium, where $(2h_{\text{EQM}}^2 - r^2) > 0$, this leads to an expression of the form $F = -ky$ (where this latter *k* is a spring constant, not to be confused with Coulomb's constant in the earlier force expressions in this example) that can be used to determine the period of oscillations of the dust particle. For the unstable equilibrium, $(2h_{\text{EQM}}^2 - r^2) < 0$ and the force law becomes $F = +kx$ and no oscillations occur; the dust particle is pushed farther and farther from equilibrium instead.

Summary:

I have outlined three general techniques here that can be extremely useful for solving physics problems:

(1) Dimensional analysis
Every equation in physics must have consistent units. This obvious but powerful statement allows us to spot mistakes and, in some cases, to guess the correct form of an equation based on the units of the variables it will depend on.

(2) Oscillations
The period of almost any oscillating system can be expressed as $T = 2\pi\sqrt{m/k}$ if we can determine a k value either from a force law like $F = -kx$ or a potential energy of the form $U = \frac{1}{2}kx^2$.

(3) Approximations
Often we are willing to trade an exact solution for an approximate one if it leads to a simplification in an equation. In particular, the binomial approximation $(1+x)^n \cong 1+nx$, valid when $|x| << 1$, is a very useful thing to know.

Bonus:

The following Fox Trot cartoon contains an error in one of the equations. Can you spot it?