Solution: 2. Mechanical Blackbox: a cylinder with a ball inside

In order to be able to calculate the required values in i, ii, iii, we need to know:
   a. the position of the centre of mass of the tubing plus particle (object) which depends on
      \( z, m, M \)
   b. the moment of inertia of the above.

The position of the CM may be found by balancing. The \( I_{CM} \) can be calculated from the period of oscillation of the tubing plus object.

Analytical steps to select parameters for plotting

I. \[ x_{CM} = \frac{mz + M \left( L/2 \right)}{m + M} \] \hspace{1cm} \text{.......................... (1)} \]

\( L \) is readily obtainable with a ruler.

\( x_{CM} \) is determined by balancing the tubing and object.
II. For small-amplitude oscillation about any point O the period $T$ is given by considering the equation:

$$\{ (M+m)R^2 + I_{CM} \} \dot{\theta} = -g(M+m)R \sin \theta \approx -g(M+m)R \theta \quad \text{ .................. (2)}$$

$$T = 2\pi \sqrt{\frac{I_{CM} + (M+m)R^2}{g(M+m)R}} \quad \text{ .................. (3)}$$

where

$$I_{CM} = \frac{1}{3} M \left( \frac{L}{2} \right)^2 + M \left( x_{CM} - \frac{L}{2} \right)^2 + m(z-x_{CM})^2$$

$$= \frac{1}{3} ML^2 + Mx_{CM}^2 - MLx_{CM} + m(z-x_{CM})^2 \quad \text{ .................. (4)}$$

Note that

$$T^2 \frac{g(M+m)}{4\pi^2} = \frac{I_{CM}}{R} + (M+m)R \quad \text{ .................. (5)}$$

**Method (a): (linear graph method)**

The equation (5) may be put in the form:

$$T^2 R = \left( \frac{4\pi^2}{g} \right) R^2 + \frac{4\pi^2 I_{CM}}{(M+m)g} \quad \text{ .................. (6)}$$

Hence the plot of $T^2 R$ v.s. $R^2$ will yield the straight line whose

Slope $\alpha = \frac{4\pi^2}{g} \quad \text{ .................. (7)}$

and y-intercept $\beta = \frac{4\pi^2 I_{CM}}{(M+m)g} \quad \text{ .................. (8)}$

Hence, $I_{CM} = \frac{(M+m)\beta}{\alpha} \quad \text{ .................. (9)}$

The value of $g$ is from equation (7): $g = \frac{4\pi^2}{\alpha} \quad \text{ .................. (10)}$
Method (b): minimum point curve method

The equation (5) implies that $T$ has a minimum value at

$$R = R_{\text{min}} = \sqrt{\frac{I_{CM}}{M + m}}$$

………………… (11)

Hence $R_{\text{min}}$ can be obtained from the graph $T$ v.s. $R$.

And therefore

$$I_{CM} = (M + m)R_{\text{min}}^2$$

………………… (12)

This equation (12) together with equation (1) will allow us to calculate the required values $z$ and $M/m$.

At the value $R = R_{\text{min}}$ equation (5) becomes

$$T_{\text{min}}^2 \frac{g(M + m)}{4\pi^2} = (M + m)R_{\text{min}} + (M + m)R_{\text{min}}$$

$$g = \frac{2R_{\text{min}}^2 \times 4\pi^2}{T_{\text{min}}^2} = \frac{8\pi^2 R_{\text{min}}^2}{T_{\text{min}}^2}$$

………………… (13)

from which $g$ can be calculated.
Results

\[ L = 30.0 \text{ cm } \pm 0.1 \text{ cm} \]

\[ x_{CM} = 17.8 \text{ cm } \pm 0.1 \text{ cm} \text{ (from top)} \]

<table>
<thead>
<tr>
<th>( x_{CM} - R ) (cm)</th>
<th>time (s) for 20 cycles</th>
<th>( T ) (s)</th>
<th>( R ) (cm)</th>
<th>( R^2 ) (cm(^2))</th>
<th>( T^2 R ) (s(^2)cm)</th>
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</thead>
<tbody>
<tr>
<td>1.1</td>
<td>18.59 18.78 18.59</td>
<td>0.933</td>
<td>16.7</td>
<td>278.9</td>
<td>14.53</td>
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<tr>
<td>2.1</td>
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<tr>
<td>3.1</td>
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<td>4.1</td>
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<td>8.1</td>
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<td>94.1</td>
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<td>1.338</td>
<td>1.7</td>
<td>2.9</td>
<td>3.04</td>
</tr>
</tbody>
</table>

Notes: at \( x_{CM} - R = 15.1, 16.1 \text{ cm}, \text{ times for 10 cycles.} \)
Method (a)

Calculation from straight line graph: slope \( \alpha = 0.04108 \pm 0.0007 \text{s}^2/\text{cm}, \) y-intercept \( \beta = 3.10 \pm 0.05 \text{s}^2\text{cm} \)

\[
g = \frac{4\pi^2}{\alpha} \text{ giving } g = (961 \pm 20) \text{ cm/s}^2
\]

\[
\frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2 (\pm 2.5 \text{ cm}^2)
\]

\[
I_{CM} = (M + m) \frac{\beta}{\alpha} = (75.46)(M + m)
\]

From equation (4): \( I_{CM} = \frac{1}{3} M \left( \frac{L}{2} \right)^2 + M \left( x_{CM} - \frac{L}{2} \right)^2 + m(z - x_{CM})^2 \)
Then \((75.46)(M + m) = 75.0M + 7.84M + m(z - 17.8)^2\)

\[-7.38 \frac{M}{m} + 75.46 = (z - 17.8)^2\]

......................... (14)

The centre of mass position gives:

\[17.8(M + m) = 15.0M + mz\]

\[\frac{M}{m} = \frac{z - 17.8}{2.8}\]

......................... (15)

From equations (14) and (15):

\[-7.38 \frac{2.8}{M} + 75.46 = (z - 17.8)^2\]

\[(z - 17.8) = 7.47\]

And \[z = 25.27 = 25.3 \pm 0.1 \text{ cm}\]

\[\frac{M}{m} = 2.68 = 2.7\]

**Error Estimation**

Find error for \(g\):

From (10), \[g = \frac{4\pi^2}{\alpha}\]

\[\Delta g = \frac{\Delta \alpha}{\alpha} g = 16.3 \text{ cm/s}^2 \approx 20 \text{ cm/s}^2\]

i) Find error for \(z\):

First, find error for \[r = \frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2\].

\[\Delta r = \left(\frac{\Delta \alpha}{\alpha} + \frac{\Delta \beta}{\beta}\right)r = 2.5 \text{ cm}^2\]

Since error from \(r\) contributes most \(\left(\frac{\Delta r}{r} \approx 0.03\right.\) while \(\frac{\Delta L}{L}, \frac{\Delta x_{cm}}{x_{cm}} \approx 0.005\), we estimate error propagation from \(r\) only to simplify the analysis by substituting the min and max values into equation (4).

Now, we use \(r_{\text{max}} = r + \Delta r = 75.46 + 2.5 = 77.96\). The corresponding quadratic equation is
\[(z-17.8)^2 + 1.743(z-17.8) - 77.96 = 0\] The corresponding solution is \((z-17.8)_{\text{max}} = 7.55\) cm

If we use \(r_{\text{min}} = r - \Delta r = 75.46 - 2.5 = 72.96\), the corresponding quadratic equation is
\[(z-17.8)^2 + 3.529(z-17.8) - 72.96 = 0\]

The corresponding solution is \((z-17.8)_{\text{min}} = 6.96\) cm

So \(\Delta(z-17.8) = \frac{7.55 - 6.96}{2} = 0.3\) cm

Note that \(\frac{\Delta(z-17.8)}{z-17.8} \approx 0.04\). So, we still ignore the error propagation due to \(\Delta L, \Delta x_{cm}\).

The error \(\Delta z\) can be estimated from \(\Delta z \approx \Delta(z-17.8) = 0.3\) cm

ii) Find error for \(\frac{M}{m}\):

We know that \(\frac{M}{m} = \frac{z-17.8}{2.8}\)

\[\Delta\left(\frac{M}{m}\right) = \frac{\Delta(z-17.8)}{2.8} = 0.11\]
**Method (b)**

Calculation from \( T-R \) plot:

\[
T = T_{\text{min}} \quad \text{at} \quad I_{CM} = (M + m)R_{\text{min}}^2 \quad \text{and} \quad g = \frac{8\pi^2 R_{\text{min}}}{T_{\text{min}}^2}
\]

Using the minimum position: \( T = T_{\text{min}} \) at \( I_{CM} = (M + m)R_{\text{min}}^2 \) and \( g = \frac{8\pi^2 R_{\text{min}}}{T_{\text{min}}^2} \)

From graph: \( R_{\text{min}} = 8.9 \pm 0.2 \) cm and \( T_{\text{min}} = 0.846 \pm 0.005 \) s

\[
\therefore \quad g = 982 \pm 40 \text{ cm/s}^2
\]

\[
I_{CM} = (M + m)(8.9)^2 = (79.21)(M + m) \quad \text{.........................} \quad (16)
\]
From equations (14), (15), (16):

\[(79.21)(M + m) = 75.0M + 7.84M + m(z-17.8)^2\]
\[-3.63M + 79.21m = m(z-17.8)^2\]
\[(x-17.8)^2 + \frac{3.63}{2.8}(x-17.8)-79.21 = 0\]
\[(z-17.8) = 8.28\]

And \[z = 26.08 = 26.1 \pm 0.7 \text{ cm}\]
\[
\frac{M}{m} = 2.95 = 3.0 \pm 0.3
\]

**Error estimation**

i) Find error for \(g\):

Using the minimum position: \(g = \frac{8\pi^2 R_{\text{min}}}{T_{\text{min}}^2}\), we have

\[
\Delta g = \left(\frac{\Delta R_{\text{min}}}{R_{\text{min}}} + 2 \frac{\Delta T_{\text{min}}}{T_{\text{min}}}\right)g = 34 \approx 30 \text{ cm/s}^2
\]

ii) Find error for \(z\):

First, find error for \(r = R_{\text{min}}^2 = 79.21 \text{ cm}^2\).

\[
\Delta r = 2 R_{\text{min}} \Delta R_{\text{min}} = 3.56 \text{ cm}^2
\]
This \(r\) is equivalent to \(r\) in part 1. So, one can follow the same error analysis.
As a result, we have
\[z = 26.08 \approx 26.1 \text{ cm}\]
\[
\Delta z = 0.8 \text{ cm}
\]

i) Find error for \(M/m\):

Following the same analysis as in part I, we found that
\[
M/m = 2.96; \quad \Delta(M/m) = 0.15
\]

NOTE: This minimum curve method is not as accurate as the usual straight line graph.