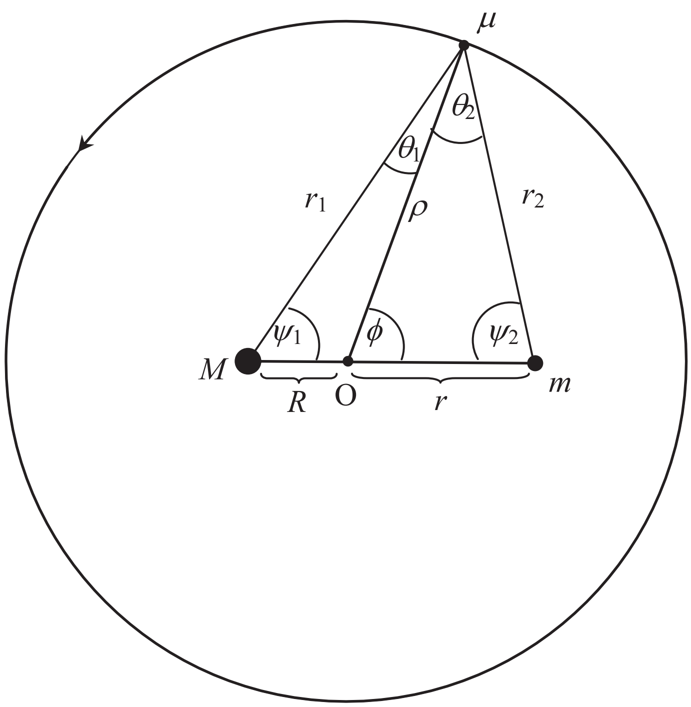
I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \quad \text{.......................... (1)}$$

$$m\omega_0^2 r = \frac{GMm}{(R+r)^2}$$

$$M\omega_0^2 R = \frac{GMm}{(R+r)^2} \quad \text{.......................... (2)}$$

From Eq. (2), or using reduced mass, $\omega_0^2 = \dfrac{G(M+m)}{(R+r)^3}$

Hence, $\omega_0^2 = \dfrac{G(M+m)}{(R+r)^3} = \dfrac{GM}{r(R+r)^2} = \dfrac{Gm}{R(R+r)^2}$. .... (3)

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m. For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1+\frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{..........................} \quad (4)$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{..........................} \quad (5)$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2=\sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \qquad \text{..........................} \quad (6)$$

The distances r_2 and ρ, the angles θ_1 and θ_2 are related by two SineRule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$
$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r} \qquad \text{..........................} \quad (7)$$

Substitute (7) into (6)

$$\frac{1}{r_2^3}=\frac{R}{(R+r)^4}\frac{(M+m)}{m} \qquad \text{..........................} \quad (10)$$

Since $\frac{m}{M+m}=\frac{R}{R+r}$,Eq. (10) gives

$$r_2 = R+r \qquad \text{..........................} \quad (11)$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R+r \qquad \text{..........................} \quad (12)$$

Alternatively,
$$\frac{r_1}{\sin(180^\circ-\phi)} = \frac{R}{\sin\theta_1} \quad \text{and} \quad \frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\psi_1 = 60^\circ \qquad \psi_2 = 60^\circ \qquad \text{.......................... (13)}$$

The distance ρ is calculated from the Cosine Rule.

$$\rho^2 = r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ$$
$$\rho = \sqrt{r^2 + rR + R^2} \qquad \text{.......................... (14)}$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.......................... (4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.......................... (5)}$$

Note that

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$$

$$\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2} \qquad \text{(see figure)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1} \qquad \text{.......................... (6)}$$

Equations (5) and (6):

$$r_1 = r_2 \qquad \text{.......................... (7)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \qquad \text{.......................... (8)}$$

$$\psi_1 = \psi_2 \qquad \text{.......................... (9)}$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3}r_1^2\rho \qquad \text{.......................... (10)}$$

Equations (8) and (10): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2\rho}{(R+r)^3}\sin\theta_2$ ……………………… (11)

Note that from figure, $\dfrac{\rho}{\sin\psi_2} = \dfrac{r}{\sin\theta_2}$ ……………………… (12)

Equations (11) and (12): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2 r}{(R+r)^3}\sin\psi_2$ ……………………… (13)

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2)+r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \quad \text{……………… (14)}$$

Equations (13) and (14): $\sin(\theta_1+\theta_2) = \dfrac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]}$ ……………………… (15)

$$\theta_1+\theta_2 = 180^\circ-\psi_1-\psi_2 = 180^\circ-2\psi_2 \quad \text{(see figure)}$$

$$\therefore \quad \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is $\rho = \sqrt{\left(\dfrac{R+r}{2}-R\right)^2+\left\{(R+r)\dfrac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$

1.3 The energy of the mass μ is given by

$$E=-\frac{GM\mu}{r_1}-\frac{Gm\mu}{r_2}+\tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2+\rho^2\omega^2\right) \quad \text{………………………..(15)}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1=r_2=\Re$ and $m=M$),

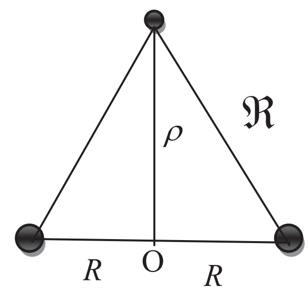
$$E=-\frac{2GM\mu}{\Re}+\tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2+\frac{\rho_0^4\omega_0^2}{\rho^2}\right)$$

………………………..(16)

Since the energy is conserved,

$$\frac{dE}{dt}=0$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^2}\frac{d\Re}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}\frac{d\rho}{dt} = 0 \text{ ....(17)}$$



$$\frac{d\Re}{dt} = \frac{d\Re}{d\rho}\frac{d\rho}{dt} = \frac{d\rho}{dt}\frac{\rho}{\Re} \text{ ....(18)}$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^3}\rho\frac{d\rho}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}\frac{d\rho}{dt} = 0 \text{ ....(19)}$$

Since $\frac{d\rho}{dt} \neq 0$, we have

$\frac{2GM}{\Re^3}\rho + \frac{d^2\rho}{dt^2} - \frac{\rho_0{}^4\omega_0{}^2}{\rho^3} = 0$ or

$$\frac{d^2\rho}{dt^2} = -\frac{2GM}{\Re^3}\rho + \frac{\rho_0{}^4\omega_0{}^2}{\rho^3}. \text{ ....(20)}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re = \Re_0\left(1 + \frac{\Delta\Re}{\Re_0}\right)$ and $\rho = \rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dt^2}(\rho_0 + \Delta\rho) = -\frac{2GM}{\Re_0^3\left(1 + \frac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right) + \frac{\rho_0{}^4\omega_0{}^2}{\rho_0^3\left(1 + \frac{\Delta\rho}{\rho_0}\right)^3} \text{ ....(21)}$$

Using binomial expansion $(1+\varepsilon)^n \approx 1 + n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right)\left(1 - \frac{3\Delta\Re}{\Re_0}\right) + \rho_0\omega_0^2\left(1 - \frac{3\Delta\rho}{\rho_0}\right). \text{ ....(22)}$$

Using $\Delta\rho = \frac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \rho_0\omega_0^2\left(1 - \frac{3\Delta\rho}{\rho_0}\right). \text{ ....(23)}$$

Since $\omega_0^2 = \frac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(1 + \frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \omega_0^2\rho_0\left(1 - \frac{3\Delta\rho}{\rho_0}\right) \text{ ....(24)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \text{ ....(25)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2 \Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \qquad \text{..................(26)}$$

From the figure, $\rho_0 = \Re_0 \cos 30°$ or $\frac{\rho_0^{\ 2}}{\Re_0^{\ 2}} = \frac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2 \Delta\rho\left(4 - \frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho\,. \qquad \text{..................(27)}$$

Angular frequency of oscillationis $\frac{\sqrt{7}}{2}\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^{\ 2} = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R + \zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2nd law, $-\frac{2GM\mu}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R+\zeta) - \mu\omega^2(\sqrt{3}R+\zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R+\zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^{\ 2}\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2 + 2\sqrt{3}\zeta R\}^{3/2}}(\sqrt{3}R+\zeta) \approx \frac{d^2\zeta}{dt^2} - \frac{\omega_0^{\ 2}\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}} = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^{\ 2}\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\omega_0^{\ 2}\sqrt{3}R\left(1 - \frac{3\sqrt{3}\zeta}{4R}\right)\left(1 + \frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2\zeta}{dt^2} - \omega_0^{\ 2}\sqrt{3}R\left(1 - \frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta = -\left(\frac{7}{4}\omega_0^{\ 2}\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, v_{BA} is the velocity of B as observed by A.

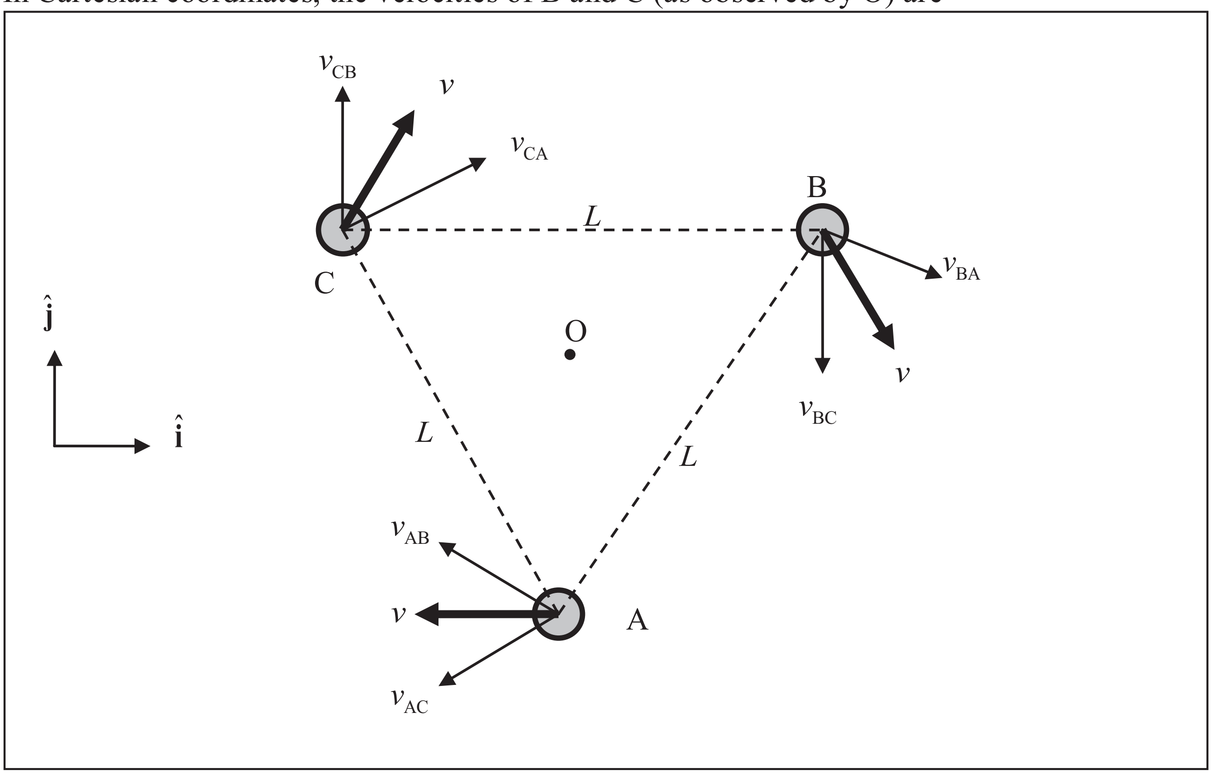
The period of circular motion is 1 year $T = 365 \times 24 \times 60 \times 60$ s. ………… (28)

The angular frequency $\omega = \dfrac{2\pi}{T}$

The speed $v = \omega \dfrac{L}{2\cos 30°} = 575$ m/s ………… (29)

The speed is much less than the speed light → Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60°\hat{\mathbf{i}} - v\sin 60°\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60°\hat{\mathbf{i}} + v\sin 60°\hat{\mathbf{j}}$

Hence $\vec{v}_{\mathrm{BC}} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s ………… (30)

Notice that the relative velocities for each pair are anti-parallel.

Alternative solution for 1.4

One can obtain v_{BC} by considering the rotation about the axis at one of the spacecrafts.

$$v_{\mathrm{BC}} = \omega L = \frac{2\pi}{365\times 24\times 60\times 60\ \mathrm{s}}(5\times 10^6\ \mathrm{km}) \approx 996\ \mathrm{m/s}$$